A note on bisections of graphs with girth at least 6

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Abstract. A *bisection* of a graph is a partition of its vertex set into two sets which differ in size by at most 1, and its *size* is the number of edges between the two sets. Let G be a graph with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees, then there is a constant $c > 0$ such that G admits a bisection of size at least $m/2 + c\sum_{i=1}^{n}\sqrt{d_i}$. It follows as a corollary that G has a bisection of size at least $m/2 + cm^{3/4}$.

Keywords: bisection, girth, perfect matching.

1. Introduction

All graphs considered here are finite, undirected and have no loops and no parallel edges, unless otherwise indicated. Given a graph $G = (V, E)$, the Max-Cut problem asks for a partition of V into V_1 and V_2, which maximizes the number of crossing edges between V_1 and V_2. This problem is NP-hard even when restricted to triangle-free cubic graphs and has been a very active research subject in both Combinatorics and Computer Science (see [8, 9, 15]).

It is easy to see by considering a random partition that every graph with m edges contains a cut of size at least $m/2$ edges. Edwards [5, 6] improved this lower bound to

$$\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8},$$

which is essentially best possible as evidenced by the complete graphs K_{2n+1}. Moreover, for certain range of m, Alon gave an additive improvement of order $m^{1/4}$.

Many results concerned on Max-Cut are about graphs with forbidden cycles. Let H be a graph, we say G is H-free if G contains no copy of H. After a series of papers by various researchers, Alon [1] showed that every triangle-free graph

G with m edges has a cut of size at least $m/2 + \Omega(m^{4/5})$. In [3], Alon *et al.* proved that for every odd integer $r > 5$, there is a $c(r) > 0$ such that every C_{r-1}-free graph with m edges has a bipartite cut of size at least

$$\frac{m}{2} + c(r)m^{r/(r+1)}.$$

It is further shown that this is tight, up to the value of $c(r)$, for $r \in \{5, 7, 11\}$. For more information on the Max-Cut problems in H-free graphs, we refer the reader to [1, 2, 3, 4, 12, 13] and their references.

In this note we focus on bisections of graphs. Given a graph $G = (V, E)$, the Max-Bisection problem is to find a bipartition of its vertex set $V = V_1 \cup V_2$, which maximizes the number of crossing edges, with the additional constraint that $||V_1| - |V_2|| \leq 1$. Indeed, the Max-Bisection problems tend to be much more complicated to analyze than the Max-Cut problems. As noticed in [7, 11], the Edwards bound implicitly implies that a connected graph G with n vertices and m edges admits a cut of size

$$\frac{m}{2} + \frac{n-1}{4}.$$

However, every bisection of $K_{1,n-1}$ has size $n/2$. Lee *et al.* [10] proved that every connected graph G with n vertices and m edges has a bisection of size at least

$$\frac{m}{2} + \frac{n+1-\Delta(G)}{4},$$

and the bound is tight.

Let $d \geq 1$ be an integer and n be sufficiently large. It is easy to see that the size of any bisection of a complete bipartite graph $K_{d,n}$ with m edges is at most $m/2 + O(1)$. Thus, we consider graphs with perfect matchings throughout the paper. Our main new results are as follows.

Theorem 1.1. *Let G be a graph on n vertices with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees, then there is a constant $c > 0$ such that G admits a bisection of size at least*

$$\frac{m}{2} + c\sum_{i=1}^{n} \sqrt{d_i}.$$

Corollary 1.2. *Let c be a positive constant. Every graph with m edges, perfect matchings and girth at least 6, has a bisection of size at least*

$$\frac{m}{2} + cm^{3/4}.$$

2. Proofs

Let $G = (V, E)$ be a graph on n vertices with m edges, perfect matchings and girth at least 6. Let $d_1, d_2, \ldots, d_n$ be vertex degrees. Suppose that $M = \{e_1, e_2, \ldots, e_{\frac{n}{2}}\}$ is a perfect matching in G. Consider the following randomized procedure for obtaining a bisection of G. Let $h : V \mapsto \{0, 1\}$ be a random function obtained by separating each $e_i = uu' \in M$ randomly and independently, either $h(u) = 0$ and $h(u') = 1$, or $h(u) = 1$ and $h(u') = 0$, where both choices are equally likely. Let $n_0(u)$ be the number of neighbours v of u satisfying $h(v) \neq h(u)$ where $v \neq u'$. Let $n_1(u)$ be the number of neighbours w of u satisfying $h(w) = h(u)$ where $w \neq u'$. Similarly let $n_0(u')$ be the number of neighbours v' of u' satisfying $h(v') \neq h(u')$ where $v' \neq u$, and $n_1(u')$ be the number of neighbours w' of u' satisfying $h(w') = h(u')$ where $w' \neq u$. Call an edge $uu' \in M$ *good* if $n_0(u) + n_0(u') > n_1(u) + n_1(u')$. If $n_0(u) + n_0(u') = n_1(u) + n_1(u')$, then call it *good with probability 1/2*. Otherwise call it *bad*. Let $h' : V \mapsto \{0, 1\}$ be the randomly function obtained from h as follows. For each $uu' \in M$, if uu' is *bad*, then choose randomly again, either $h'(u) = 0$ and $h'(u') = 1$, or $h'(u) = 1$ and $h'(u') = 0$, where both choices are equally likely and all choices are independent. Otherwise define $h'(u) = h(u)$ and $h'(u') = h(u')$. Finally, define $V_0 = (h')^{-1}(0)$ and $V_1 = (h')^{-1}(1)$. We obtain a bisection $V = V_0 \cup V_1$ and we denote each $uu' \in M$ by e_u (or $e_{u'}$).

We wish to calculate the expected size of this bisection. We want the probability that each edge ends up in the bisection. Note that for each $uu' \in M$, $\mathbb{P}[h'(u) \neq h'(u')] = 1$. Hence we aim to determine this probability for every edge in $E - M$.

Lemma 2.1. *Let G be a graph with perfect matchings. Let M be a perfect matching in G, and let $h, h' : V \mapsto \{0, 1\}$ be the two random functions defined by the randomized procedure described above. For a given edge $uv \in E - M$, let $p = \mathbb{P}[e_u$ and e_v are good $\mid h(u) = h(v)], q = \mathbb{P}[e_u$ and e_v are good $\mid h(u) \neq h(v)]$. Then*

$$\mathbb{P}[h'(u) \neq h'(v)] = \frac{1}{2} + \frac{1}{4}(q - p).$$

Proof. Let uv be an edge in $E - M$. By the law of total probability,

$$\begin{aligned} \mathbb{P}[h'(u) \neq h'(v)] &= \frac{1}{2}\mathbb{P}[h'(u) \neq h'(v) \mid h(u) = h(v)] \\ &\quad + \frac{1}{2}\mathbb{P}[h'(u) \neq h'(v) \mid h(u) \neq h(v)]. \end{aligned} \tag{1}$$

Clearly, if $h(u) = h(v)$ and at least one of the edges e_u or e_v is bad, then the probability that $h'(u) \neq h'(v)$ is 1/2. But if both e_u and e_v are good, then the probability that $h'(u) \neq h'(v)$ is 0. Thus the conditional probability

$$\mathbb{P}[h'(u) \neq h'(v) \mid h(u) = h(v)] = 0 \cdot p + \frac{1}{2} \cdot (1 - p) = \frac{1}{2}(1 - p). \tag{2}$$

Similarly, if $h(u) \neq h(v)$ and at least one of the edges e_u or e_v is bad, then the probability that $h'(u) \neq h'(v)$ is $1/2$, whereas if they are both good, then certainly $h'(u) \neq h'(v)$. This yields that

$$\mathbb{P}[h'(u) \neq h'(v) \mid h(u) \neq h(v)] = 1 \cdot q + \frac{1}{2} \cdot (1-q) = \frac{1}{2}(1+q). \tag{3}$$

Combining (1), (2) and (3), we get the required result. □

It remains to estimate the difference $q-p$. Let $p_u = \mathbb{P}[e_u \text{ is good } \mid h(u) = h(v)], p_v = \mathbb{P}[e_v \text{ is good } \mid h(u) = h(v)]$. Similarly, let $q_u = \mathbb{P}[e_u \text{ is good } \mid h(u) \neq h(v)]$ and $q_v = \mathbb{P}[e_v \text{ is good } \mid h(u) \neq h(v)]$. For each vertex u of G, let d_u denote the degree of u in G. We then have the following lemma.

Lemma 2.2. *Let G be a graph with perfect matchings and girth at least 6. Let M be a perfect matching in G, and let $h, h' : V \mapsto \{0,1\}$ be the two random functions defined by the randomized procedure described above. For a given edge $uv \in E - M$,*

$$\mathbb{P}\big[h'(u) \neq h'(v)\big] = \frac{1}{2} + \frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right).$$

Proof. Since G is a graph with girth at least 6, the event e_u is good given $h(u) = h(v)$ is independent of the event e_v is good given $h(u) = h(v)$. Then $p = p_u p_v$. Similarly, $q = q_u q_v$. Let uu' and vv' be the edges in M. Because of the way we defined good and bad, if $d_u + d'_u$ is odd, then

$$\begin{aligned}\mathbb{P}[e_u \text{ is good }] = \left(\frac{1}{2}\right)^{d_u+d'_u-2} &\left(\binom{d_u+d'_u-2}{0} + \binom{d_u+d'_u-2}{1}\right.\\ &\left. + \cdots + \binom{d_u+d'_u-2}{(d_u+d'_u-3)/2}\right) = \frac{1}{2}.\end{aligned}$$

The computation for the case that $d_u + d'_u$ is even is similar. Hence we have $\mathbb{P}[e_u \text{ is good }] = 1/2$. Similarly, $\mathbb{P}[e_v \text{ is good }] = 1/2$. It follows that $p_u + p_v = 1$, $q_u + q_v = 1$. This, together with Lemma 2.1, implies that

$$\mathbb{P}\big[h'(u) \neq h'(v)\big] = \frac{1}{2} + \frac{1}{4}(p_u p_v - q_u q_v) = \frac{1}{2} + \frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right).$$

This completes the proof of Lemma 2.2. □

Next we employ a result of Shearer [12], which shows that the truth of Theorem 1.1 implies the truth of Corollary 1.2.

Lemma 2.3 ([12])**.** *For any graph G with m edges and vertex degrees $d_1, d_2, \ldots, d_n$, we have $\sum_{i=1}^{n} \sqrt{d_i} \geq m^{3/4}$.*

Finally, we shall also need two simple facts. The first one can be found in [14].

Lemma 2.4 ([14]). *Let δ be a positive constant. Every graph on n vertices with m edges and perfect matchings, has a bisection of size at least $m/2 + \delta n$.*

Lemma 2.5. *Let G be a graph on n vertices with m edges and perfect matchings. Suppose that M is a perfect matching in G and $M' \subseteq M$. Let the vertex set $X = V(M')$ and $G' = G - X$ be a subgraph of G with m' edges. If G' has a bisection of size k, then G has a bisection of size at least $k + (m - m')/2$.*

Proof. Let $G = (V, E)$ be a graph on n vertices with m edges, and $G' = (V', E') = G - X$ be a subgraph of G with m' edges. Let $M' = \{u_1v_1, \cdots, u_tv_t\}$, $1 \le t \le n/2$. Suppose G' admits a bisection $V' = V_1 \cup V_2$ of size k. We now get a bisection of G by adding each pair $\{u_i, v_i\}$ in X into V' in turn, either $u_i \in V_1$ and $v_i \in V_2$, or $u_i \in V_2$ and $v_i \in V_1$, trying to maximize the number of crossing edges in the bisection. This immediately implies that there is a bisection in G which contains all the k edges of the initial bisection of G', and at least half of all other edges. This completes the proof. □

Having finished all the necessary preparations we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $G = (V, E)$ be a given graph and let M be a perfect matching in G. As long as there is an edge $uu' \in M$ of degrees $d_u + d_{u'} \le 3$, delete the two vertices $\{u, u'\}$. If during this process we delete more than $\frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ vertices, by Lemma 2.4, we get the desired result, with $c = \frac{\delta}{4}$ and where δ is the constant from that lemma. Therefore we may assume that the process terminates after deleting at most $\frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ vertices. It thus terminates with an induced subgraph $G' = (V', E')$ with n' vertices, m' edges and degree sequence $d'_1, \dots, d'_n$. Clearly, G' also has a perfect matching $M' \subseteq M$, in which every edge uu' satisfies the condition that the degree of u in G' plus the degree of u' in G' is more than 3. Note that G' is obtained from G by deleting fewer than $2 \cdot \frac{1}{8}\sum_{i=1}^{n}\sqrt{d_i} = \frac{1}{4}\sum_{i=1}^{n}\sqrt{d_i}$ edges. Since each deletion of an edge decreases the sum of roots of the degrees by at most 2, we have

$$\sum_{i=1}^{n'}\sqrt{d'_i} \ge \frac{1}{2}\sum_{i=1}^{n}\sqrt{d_i}. \tag{4}$$

Let uv be an edge of G'. Suppose that uu', vv' are edges in M'. Consider the randomized procedure for obtaining a bisection of G' by choosing the random fuctions $h, h' : V' \mapsto \{0, 1\}$ as described in the beginning of the section. Clearly p_u depends only on the degree d of u in G' and the degree d' of u' in G'. By Lemma 2.2, the expected size of our bisection is

$$\begin{aligned}\sum_{uv\in E'}\mathbb{P}\big[h'(u) \ne h'(v)\big] &= |M'| + \sum_{uv\in E'-M'}\mathbb{P}\big[h'(u) \ne h'(v)\big]\\ &= \frac{m}{2} + \frac{|M'|}{2} + \sum_{uv\in E'-M'}\left(\frac{1}{4}\left(p_u - \frac{1}{2}\right) + \frac{1}{4}\left(p_v - \frac{1}{2}\right)\right)\end{aligned}$$

$$= \frac{m'}{2} + \frac{|M'|}{2} + \frac{1}{4}\sum_{u\in V'}(d-1)\left(p_u - \frac{1}{2}\right).$$

Also, because of the definition of p_u, the expected size of the bisection is

$$\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{1}{4}\sum_{u\in V'}(d-1)\cdot\frac{1}{2^{d+d'-2}}\cdot\binom{d+d'-3}{\lfloor(d+d'-3)/2\rfloor}. \tag{5}$$

Claim 2.6. *For any integer $a > 0$, there exist a constant $0 < c_1 < 1$ such that*

$$\frac{1}{2^a}\binom{a}{\lfloor a/2\rfloor} \geq \frac{c_1}{\sqrt{a}}.$$

The desired result follows from Stirling's approximation.

Claim 2.7.

$$\sqrt{d+d'-3} \geq \frac{\sqrt{2}}{2}(\sqrt{d}+\sqrt{d'}) - 3.$$

If $\sqrt{d}+\sqrt{d'} \leq 3\sqrt{2}$, then $(\sqrt{2}/2)(\sqrt{d}+\sqrt{d'}) - 3 \leq 0 \leq \sqrt{d+d'-3}$. If $\sqrt{d}+\sqrt{d'} > 3\sqrt{2}$, then

$$\left(\sqrt{d+d'-3}\right)^2 - \left[(\sqrt{2}/2)(\sqrt{d}+\sqrt{d'}) - 3\right]^2$$
$$= \frac{1}{2}\left(d+d'-2\sqrt{dd'}\right) - 12 + 3\sqrt{2}\left(\sqrt{d}+\sqrt{d'}\right) > \frac{1}{2}\left(d-d'\right)^2 + 3\sqrt{2}\cdot 3\sqrt{2} - 12 > 0.$$

This completes the proof of Claim 2.7.

Now it follows from (5) and Claim 2.6 that G' has a bisection of size at least

$$\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{1}{8}\sum_{u\in V'}(d-1)\frac{c_1}{\sqrt{d+d'-3}}. \tag{6}$$

Note that

$$\sum_{u\in V'}\frac{(d-1)}{\sqrt{d+d'-3}} = \sum_{uu'\in M'}\frac{(d+d'-2)}{\sqrt{d+d'-3}} > \sum_{uu'\in M'}\sqrt{d+d'-3} = \frac{1}{2}\sum_{u\in V'}\sqrt{d+d'-3}.$$

This together with Claim 2.7 and (6), implies that G' admits a bisection of size at least

$$\frac{m'}{2} + \lfloor\frac{n'}{4}\rfloor + \frac{c_1}{16}\cdot\frac{\sqrt{2}}{2}\left(\sum_{u\in V'}(\sqrt{d}+\sqrt{d'}) - 3n'\right)$$
$$> \frac{m'}{2} + \frac{\sqrt{2}c_1}{16}\sum_{u\in V'}\sqrt{d} + \left(\lfloor\frac{n'}{4}\rfloor - \frac{3\sqrt{2}c_1}{32}n'\right)$$
$$> \frac{m'}{2} + c\sum_{i=1}^{n'}\sqrt{d_i'}.$$

The desired result follows from (4) and Lemma 2.5 by choosing $c = \frac{\sqrt{2}}{16}c_1$. □

3. Concluding remarks

This note is motivated by the results on Max-Cuts of triangle-free graphs given by Shearer [12]. But bisection problems tend to be much more complicated to analyze than the Max-Cut problems. The proofs in Section 2 combine combinatorial and probabilistic techniques. We also define several random events in the proofs. To ensure the mutual independence of these events, we consider the graphs with girth at least 6. However, there are still very few results on the maximum bisections of graphs.

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